Sound velocity and isentropic exponents of real air on its compressibility chart

D. A. Kouremenos and K. A. Antonopoulos
National Technical University of Athens, Mechanical Engineering Department, Athens, Greece

In previous publications, three real-gas isentropic exponents k_{pv}, k_{Tv}, and k_{pT} have been introduced, which, when used instead of the classical exponent $k=c_p/c_v$ in the ideal-gas isentropic-change equations, may be applied with great accuracy to real gases. The first of the exponents may also be employed for calculating the real-gas sound velocity as $\alpha=(k_{pv}pv)^{1/2}$. The purpose of this study is to present the generalized compressibility chart (of the Nelson–Obert type) of real air and place on this chart the values of the real-air isentropic exponents and sound velocity based on its virial equation of state. The usefulness of such diagrams lies in the fact that they provide in a compact and concise form information about the thermodynamic behavior and the isentropic flow of real air, and they allow straightforward comparisons between air and other real gases. Using the Redlich–Kwong–Soave correlation with acentric factor $\omega=0.035$, similar diagrams for the air are presented and compared successfully with the corresponding exact ones.

Keywords: real air; sound velocity; isentropic exponents; compressibility chart

Introduction

In previous publications,[1–3] three isentropic exponents k_{pv}, k_{Tv}, and k_{pT} for real gases have been introduced instead of the (one) classical isentropic exponent $k=c_p/c_v$. It has been shown[1–3] that if the latter is replaced in the ideal-gas isentropic-change equations by the appropriate one of the three real-gas exponents, these equations may describe very accurately small or differential isentropic changes of real gases. Extended isentropic changes may also be calculated very accurately in a stepwise fashion by using at each step the local values of the exponents introduced.

The expressions derived[1–3] for the three real-gas exponents and the relation connecting them are

$$k_{pv}=-\frac{v}{p}\frac{c_p}{c_v}\left(\frac{\partial p}{\partial v}\right)_T \tag{1}$$

$$k_{Tv}=1+\frac{v}{c_v}\left(\frac{\partial p}{\partial T}\right)_v \tag{2}$$

$$k_{pT}=\left[1-\frac{p}{c_p}\left(\frac{\partial v}{\partial T}\right)_p\right]^{-1} \tag{3}$$

$$\frac{k_{pv}}{k_{Tv}-1}=\frac{k_{pT}}{k_{pT}-1} \tag{4}$$

where p, v, and T stand for the pressure, the specific volume, and the temperature, respectively, and c_p and c_v are the constant-pressure and constant-volume specific heats, respectively.

The sound velocity, α, is defined as

$$\alpha=\left[\left(\frac{\partial p}{\partial \rho}\right)_s\right]^{1/2}=\left[-v^2\left(\frac{\partial p}{\partial v}\right)_s\right]^{1/2} \tag{5}$$

where $\rho=1/v$ stands for the density of the gas and subscript s denotes constant entropy. For a real gas, it is

$$\left(\frac{\partial p}{\partial v}\right)_s=\frac{c_p}{c_v}\left(\frac{\partial p}{\partial v}\right)_T \tag{6}$$

and Equation 5 becomes

$$\alpha=\left[-v^2\frac{c_p}{c_v}\left(\frac{\partial p}{\partial v}\right)_T\right]=\left[pv\left[-\frac{v}{p}\frac{c_p}{c_v}\left(\frac{\partial p}{\partial v}\right)_T\right]\right]^{1/2} \tag{7}$$

By use of Equation 1, the above equation gives

$$\alpha=[pvk_{pv}]^{1/2} \tag{8}$$

i.e., the sound velocity of a real gas may be calculated in terms of the real-gas isentropic exponent k_{pv} by the above simple relation.

The values of the sound velocity and of the three real-gas isentropic exponents have been calculated in a general gas-independent reduced form in previous publications[4–6] by use of the Lee–Kesler, the Redlich–Kwong, and the Redlich–Kwong–Soave generalized equations of state.

Air, the thermodynamic properties of which are described by a virial equation of many terms (see Equation 13), will be referred to in the following as *real air*. The purpose of the present work is to calculate and present a generalized compressibility chart (of the Nelson–Obert[7,8] type) for real air and place on this chart the values of the real-air isentropic exponents and sound velocity. The calculations involved are based on the virial equation of state of real air. The usefulness of the charts developed lies in the fact that they provide, in a compact and concise form, information about the thermodynamic characteristics of the air. The values of the isentropic exponents taken from the above charts or from the developed relations may be employed not only in the field of thermodynamics but also in other fields of engineering, including gas dynamics, fluid mechanics, heat transfer, theory of internal combustion engines, etc., as described below in the section on usefulness of the real-gas exponents. Moreover, the charts presented allow comparisons to be made between the air and other real gases for the purpose of drawing generalized conclusions. Such charts for real gases of practical interest, including steam, refrigerants

Address reprint requests to Professor Kouremenos at the National Technical University of Athens, Mechanical Engineering Department, 42 Patission Street, Athens 106 82, Greece.

Received 27 February 1990; accepted 17 October 1990

R12 and R22, and ammonia are in preparation by one of the present authors.

Virial equations for real air

The equation of state of the real air as well as its isentropic exponents and sound velocity will be expressed in terms of reduced variables, i.e.,

Reduced pressure: $p_r = p/p_c$ (9)

Reduced temperature: $T_r = T/T_c$ (10)

Reduced specific volume: $v_r = (vp_c)/(RT_c)$ (11)

Compressibility factor: $z = (pv)/(RT) = p_r v_r / T_r$ (12)

where $R = 287.22$ J/kg K is the constant of the air and $p_c = 37.66$ bar, $T_c = 132.52$ K, and $v_c = 3.19$ dm^3/kg are its critical pressure, temperature, and specific volume, respectively.

The equation of state of real air in the temperature range 60 K < T < 450 K (i.e., $0.45 < T_r < 3.40$) may be expressed in terms of the reduced variables p_r, v_r, and T_r as

$$p_r(v_r, T_r) = \sum_{i=0}^{12} [A_{1i}T_r + A_{2i} + A_{3i}T_r^{-1} + A_{4i}T_r^{-2} + A_{5i}(T_r - 1)(T_r - 2)T_r^{-11}]\left[\frac{C}{v_r} - 1\right]^i \quad (13)$$

where the values of the coefficients A may be found in Baehr and Schwier[9] and where $C = (p_c v_c)/(RT_c) = 0.3161325$.

For deriving an expression of the isentropic exponent k_{pv}, reduced in the form k_{pv}/k, the derivative $(\partial p/\partial v)_T$ is calculated by differentiating the equation of state (Equation 13) and then substituting into Equation 1. The resulting expression is

$$\frac{k_{pv}}{k} = \frac{C}{p_r v_r} \sum_{i=0}^{12} i[A_{1i}T_r + A_{2i} + A_{3i}T_r^{-1} + A_{4i}T_r^{-2} + A_{5i}(T_r - 1)(T_r - 2)T_r^{-11}]\left[\frac{C}{v_r} - 1\right]^{i-1}. \quad (14)$$

Similarly, an expression for the isentropic exponent k_{Tv}, reduced in the form $(k_{Tv} - 1)/(R/c_v)$, is derived by calculating derivative $(\partial p/\partial T)_v$, which is then substituted into Equation 2, i.e.,

$$\frac{k_{Tv} - 1}{(R/c_v)} = v_r \sum_{i=0}^{12} [A_{1i} - A_{3i}T_r^{-2} - 2A_{4i}T_r^{-3} + A_{5i}(-9T_r^{-10} + 30T_r^{-11} - 22T_r^{-12})]\left[\frac{C}{v_r} - 1\right]^i. \quad (15)$$

The third isentropic exponent of real air may be calculated in terms of the other two, by use of Equation 4.

Lastly, the sound velocity of real air, reduced by the ideal-gas sound velocity $\alpha_{id} = (kRT)^{1/2}$, is calculated by using Equations 8 and 12 as

$$\frac{\alpha}{\alpha_{id}} = \frac{(k_{pv}pv)^{1/2}}{(kRT)^{1/2}} = \left(z\frac{k_{pv}}{k}\right)^{1/2} \quad (16)$$

or, because of Equations 14 and 12,

$$\frac{\alpha}{\alpha_{id}} = \left[\frac{C}{T_r}\sum_{i=0}^{12} i[A_{1i}T_r + A_{2i} + A_{3i}T_r^{-1} + A_{4i}T_r^{-2} + A_{5i}(T_r - 1)(T_r - 2)T_r^{-11}]\left[\frac{C}{v_r} - 1\right]^{i-1}\right]^{1/2} \quad (17)$$

Figure 1a shows the compressibility chart of real air, which has been plotted by using the equation of state (Equation 13) as follows: For v_r = const., T_r is varied from 1 to 2.5 and the resulting pairs of values $p_r - z$ (calculated from Equations 13 and 12, respectively) give the v_r = const. line. The T_r-contours have been plotted in an analogous fashion. In a small region near the critical point ($p_r = 1$, $T_r = 1$), there is some uncertainty about validity of the equation of state. The same remark applies to Figures 2, 3, and 4 mentioned in the following.

Lines of constant k_{pv}/k, $(k_{Tv} - 1)/(R/c_v)$, and α/α_{id} plotted on the compressibility chart by suitably employing Equations 14, 15, and 17 are shown in Figures 2a, 3a, and 4a, respectively.

Approximate equations based on the Redlich–Kwong–Soave correlation

Very good approximations for k_{pv}/k, $(k_{Tv} - 1)/(R/c_v)$, and α/α_{id} may be obtained by using the Redlich–Kwong–Soave[10,11] generalized equation of state, as described below.

The Redlich–Kwong–Soave equation is usually written in the form

$$p = \frac{RT}{v - b} - \frac{a[1 + L - L(T/T_c)^{0.5}]^2}{v(v + b)} \quad (18)$$

where

$$a = AR^2T_c^2/p_c, \qquad A = 0.42748 \quad (19)$$

$$b = BRT_c/p_c, \qquad B = 0.08664 \quad (20)$$

and L is a function of the acentric factor ω, i.e.,

Notation

A_{ij}	Coefficients in the equation of state of the air
A, B	Constants in the Redlich–Kwong–Soave equation of state, $A = 0.42748$, $B = 0.08664$
a, b	Coefficients in the Redlich–Kwong–Soave equation of state
C	Constant in the equation of state of the air, $C = 0.3161325$
c_p, c_v	Specific heat under constant pressure and constant volume, respectively
k	Classical isentropic exponent, $k = c_p/c_v$
k_{pv}, k_{Tv}, k_{pT}	Real-gas isentropic exponents corresponding to the pairs of variables (p, v), (T, v), (p, T), respectively
L	Parameter in the Redlich–Kwong–Soave equation of state
p, p_c, p_r	Pressure, critical pressure, and reduced pressure, respectively
R	Gas constant
s	Entropy
T, T_c, T_r	Temperature, critical temperature, and reduced temperature, respectively
v, v_c, v_r	Specific volume, critical specific volume, and reduced specific volume, respectively
z	Compressibility factor

Greek symbols

α, α_{id}	Sound velocity and ideal-gas sound velocity, respectively
ρ	Density
ω	Acentric factor

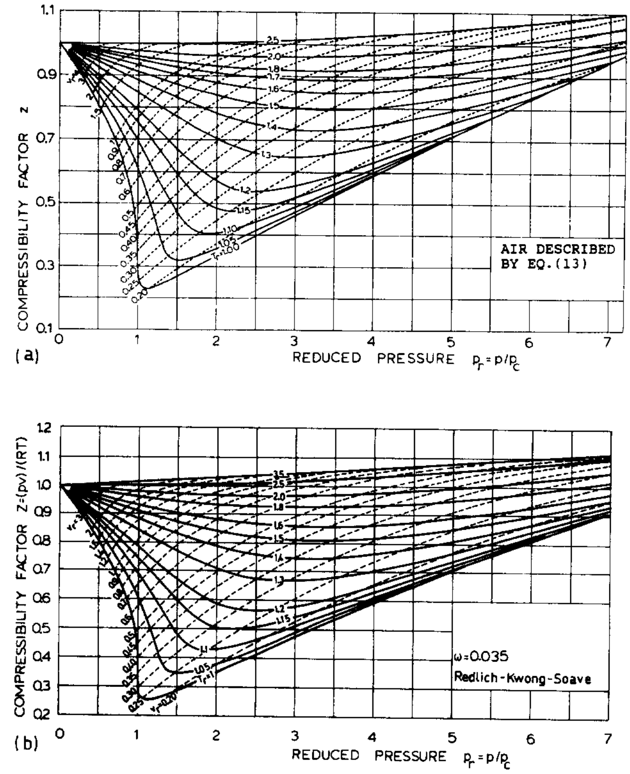

Figure 1 Compressibility chart of the air: (a) calculated by using its virial equation of state; (b) calculated by using the Redlich–Kwong–Soave correlation ($\omega = 0.035$)

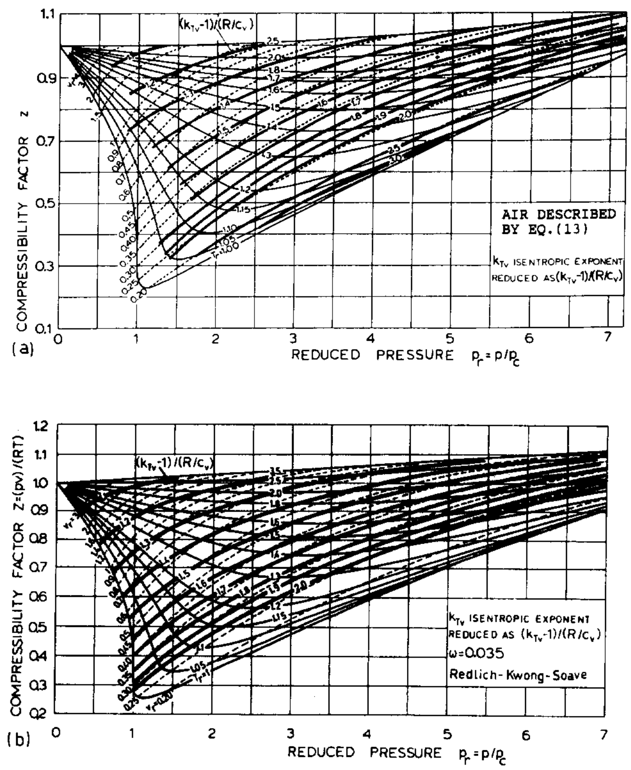

Figure 3 Lines of constant reduced isentropic exponent $(k_{Tv}-1)/(R/c_v)$ of the air: (a) calculated by using its virial equation of state; (b) calculated by using the Redlich–Kwong–Soave correlation ($\omega = 0.035$)

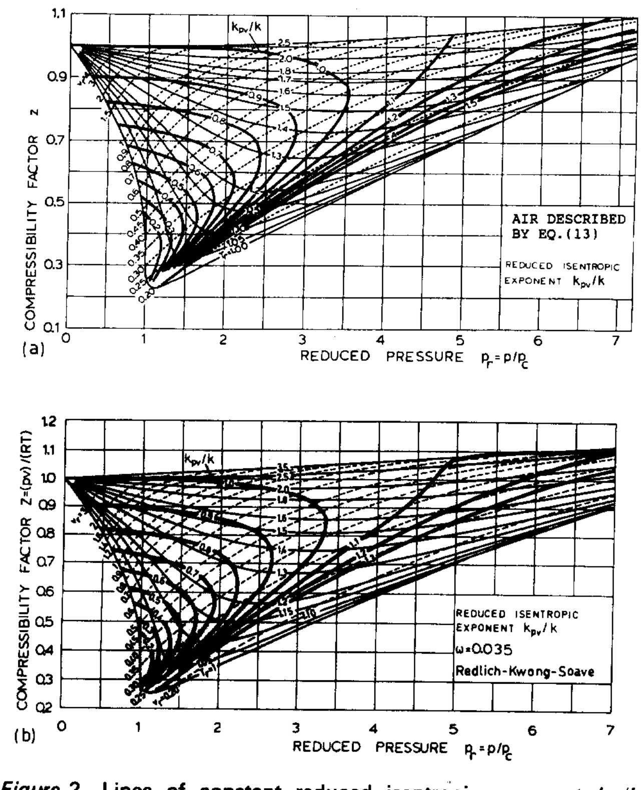

Figure 2 Lines of constant reduced isentropic exponent k_{pv}/k of the air: (a) calculated by using its virial equation of state; (b) calculated by using the Redlich–Kwong–Soave correlation ($\omega = 0.035$)

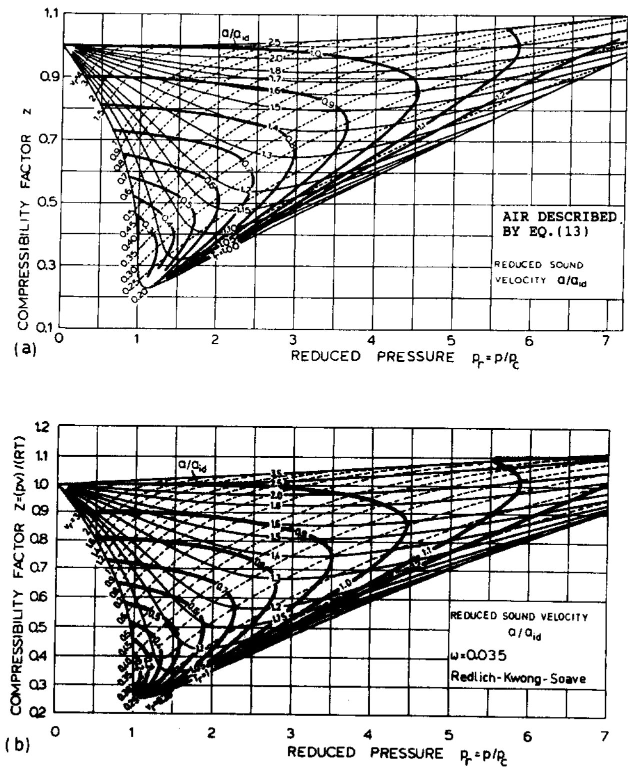

Figure 4 Lines of constant reduced sound velocity α/α_{id} of the air: (a) calculated by using its virial equation of state; (b) calculated by using the Redlich–Kwong–Soave correlation ($\omega = 0.035$)

$$L = 0.480 + 1.574\omega - 0.176\omega^2. \tag{21}$$

By employing Equations 9–12 and making suitable algebraic transformations, the Redlich–Kwong–Soave equation (Equation 18) is expressed in reduced form as

$$z = \frac{v_r}{v_r - B} - \frac{A(1 + L - LT_r^{0.5})^2}{T_r(v_r + B)} \tag{22}$$

and the following expressions are derived for the reduced isentropic exponents and sound velocity:

$$\frac{k_{pv}}{k} = \frac{1}{z}\left[\left[\frac{v_r}{v_r - B}\right]^2 - \left[\frac{v_r}{v_r - B} - z\right]\left[1 + \frac{v_r}{v_r + B}\right]\right] \tag{23}$$

$$\frac{k_{Tv} - 1}{(R/c_v)} = \frac{v_r}{v_r - B} + \frac{AL(1 + L - LT_r^{0.5})}{T_r^{0.5}(v_r + B)} \tag{24}$$

$$\frac{\alpha}{\alpha_{id}} = \left[\left[\frac{v_r}{v_r - B}\right]^2 - \left[\frac{v_r}{v_r - B} - z\right]\left[1 + \frac{v_r}{v_r + B}\right]\right]^{1/2}. \tag{25}$$

The above relations suggest that the reduced isentropic exponents and sound velocity k_{pv}/k, $(k_{Tv} - 1)/(R/c_v)$, and α/α_{id} depend only on the reduced volume and temperature, v_r and T_r, and are therefore independent of the gas properties, except for property ω contained in L and z expressions. Thus, for each value of the acentric factor ω, one compressibility chart may be plotted and the values of k_{pv}/k, $(k_{Tv} - 1)/(R/c_v)$, and α/α_{id} may be placed on this chart. Figure 1b shows the compressibility chart for $\omega = 0.035$ (air) and Figures 2b, 3b, and 4b show lines of constant k_{pv}/k, $(k_{Tv} - 1)/(R/c_v)$ and α/α_{id}, respectively, plotted on this compressibility chart. Calculation and plotting of these diagrams is not explicit, as deduced from the form of Equations 22–25. Thus, for example, for calculating a $k_{pv}/k = \text{const.}$ line, the following procedure is employed: For $k_{pv}/k = \text{const.}$ and a selected value of v_r, z is calculated explicitly from Equation 23 and T_r from Equation 22, which is of the second order in respect to $T_r^{0.5}$. With the known values of v_r, z, and T_r, the value of p_r is calculated explicitly from Equation 12, and coordinates (p_r, z) define on the compressibility chart a point lying on the $k_{pv}/k = \text{const.}$ line.

Usefulness of the real-gas exponents

As mentioned in the Introduction, the ideal-gas isentropic-change equations may describe very accurately small isentropic changes of real gases if k is replaced by the three corresponding exponents, i.e.,

$$pv^{k_{pv}} = \text{const.} \tag{26}$$

$$Tv^{(k_{Tv} - 1)} = \text{const.} \tag{27}$$

$$p^{(1 - k_{pT})}T^{k_{pT}} = \text{const.} \tag{28}$$

Therefore the exponents introduced are useful in the following cases:

(1) The exponents permit calculation of isentropic changes of real gases in cases where the entropy function $s = f(T, v)$ is not available. Small isentropic changes are calculated by Equations 26–28 with great accuracy. Extended changes may also be calculated very accurately if the calculation is performed step by step, since the values of the three exponents vary with p, v, and T according to Equations 1–3. An example for real air is given in Figure 5, which shows that the results obtained by using Equation 27 are identical to the exact solution (i.e., by the use of the entropy function $s = f(T, v) = \text{const.}$), while those obtained by the use of the classical exponent k produce an error that reaches 30%. It is worth noting that the computer time required for the calculation using Equation 27 with temperature steps of 2 K was 300% shorter than that required for the exact solution.

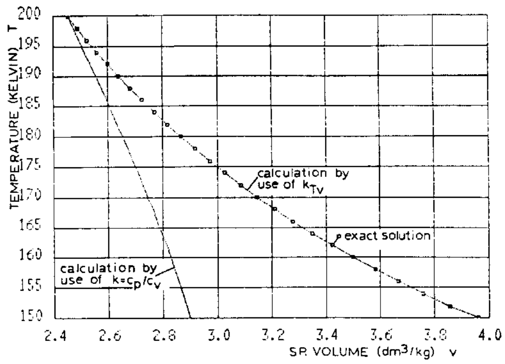

Figure 5 Isentropic expansion of real air starting at 200 bar, 2.455 dm³/kg, 200 K, and calculated (a) by the use of exponent k_{Tv}; (b) by the use of $k = c_p/c_v$; (c) by solving the entropy function $s(T, v) = 0$

(2) Equations 26–28 are useful even when the entropy equation of the real gas is available. This equation, being usually a complicated function of T and v, cannot be solved explicitly for any of these variables. Therefore, calculation, for example of v when T is known, requires a numerical iterative solution of the entropy equation $s = f(T, v)$. This is a tedious and time-consuming problem unless a good initial guess is available. Such a guess is obtained by the use of Equation 27.

(3) Apart from thermodynamics, the exponent k appears in relations encountered in gas dynamics, fluid mechanics, heat transfer, theory of internal combustion engines, etc. These relations will become more accurate if the new exponents are employed instead of k. As an example, the blow-by problem is considered. This refers to the calculation of the amount of gases flowing from the cylinder-head side of internal combustion engines or high-pressure air compressors to the crankcase side through the ring belt. Using the classical exponent k, the mass flow rate per unit area, $\dot{m}/A$, is[2]

$$\frac{\dot{m}}{A} = \left[\frac{p_1}{v_1}\right]^{1/2}\left[\frac{2}{z_1}\frac{c_p}{R}\left[\left(\frac{p_2}{p_1}\right)^{2/k} - \left(\frac{p_2}{p_1}\right)^{(k+1)/k}\right]\right]^{1/2} \times \left[1 - \left(\frac{p_2}{p_1}\right)^{2/k}\right]^{-1/2} \tag{29}$$

while the corresponding expression when the new exponents are employed is

$$\frac{\dot{m}}{A} = \left[\frac{p_1}{v_1}\right]^{1/2}\left[\frac{2}{z_1}\frac{c_p}{R}\left[\left(\frac{p_2}{p_1}\right)^{2/k_{pv}} - \left(\frac{p_2}{p_1}\right)^{(k_{Tv}+1)/k_{pv}}\right]\right]^{1/2} \times \left[1 - \left(\frac{p_2}{p_1}\right)^{2/k_{pv}}\right]^{-1/2} \tag{30}$$

where subscripts 1 and 2 denote upstream and downstream conditions, respectively. Equation 29, when applied to real air, gives an error of more than 5% under usual conditions.[2] The error is eliminated by the use of Equation 30, thus showing the deviation produced by employing k instead of the new exponents.

(4) The new exponents may also be employed in the calculation of the shock wave of real gases, as described in refs. 5 and 12.

Conclusions

The charts presented provide, in a compact and concise form, information about the thermodynamic properties of real air and allow comparisons between the air and other real gases, for which similar charts may be plotted. Such charts for real gases of practical interest, including steam, refrigerants R12 and R22, and ammonia, are in preparation by one of the present authors.

The values of the real-air isentropic exponents taken from the charts or the corresponding relations developed are useful in the calculation of the real-air isentropic change. Generally, they may replace the classical isentropic exponent $k=c_p/c_v$ in relations encountered in thermodynamics, gas dynamics, fluid mechanics, heat transfer, internal combustion engines, etc., in order to make these relations more accurate, as described in the previous section.

The behavior of the ideal gas is characterized by the assumption of noninteraction between molecules. For real gases at low density, this assumption can be considered to be close to reality, and therefore the real-gas sound velocity, α, is approximately equal to the ideal-gas sound velocity, α_{id}, i.e., $\alpha/\alpha_{id}\simeq 1$, as shown in Figure 4. However, at higher densities the interaction of the real-gas molecules becomes considerable and affects their motion, thus giving sound velocities different from the ideal-gas values. Near the critical point this interaction is considerably stronger, as shown in Figure 4.

Because of Equation 16, the pattern of the k_{pv}/k contours, shown in Figure 2, is similar to that for α/α_{id}. The $(k_{Tv}-1)/(R/c_v)$ contours of Figure 3 present a mathematically derived quantity, useful in computing properties and values, with no significant physical meaning.

Comparison of Figures 2a, 3a, and 4a with 2b, 3b, and 4b, respectively, reveals that the agreement of the k_{pv}/k, $(k_{Tv}-1)/(R/c_v)$, and α/α_{id} from the virial equation of real air with the corresponding approximate values calculated by the Redlich–Kwong–Soave correlation is generally satisfactory, being very good in some regions and less so in others.

References

1 Kouremenos, D. A. and Antonopoulos, K. A. Sur l'expansion et la compression isentropique de l'air réel entre -50°C et 450°C. *Entropie*, 1985, **125/126**, 121–128

2 Kouremenos, D. A. and Antonopoulos, K. A. Isentropic exponents of real gases and application for the air at temperatures from 150 K to 450 K. *Acta Mech.*, 1986, **65**, 81–99

3 Kouremenos, D. A. and Antonopoulos, K. A. On the isentropic change of air. *J. Méc. Théor. Appl.*, 1986, **5**(3), 515–534

4 Kouremenos, D. A., Antonopoulos, K. A., and Kakatsios, X. K. A correlation of the isentropic exponents of real gases. *Int. J. Heat Fluid Flow*, 1988, **9**, 410–414

5 Kouremenos, D. A. The normal shock waves of real gases and the generalized isentropic exponents. *Forschung im Ingenieurwesen*, 1986, **52**(1), 23–31

6 Kouremenos, D. A. and Antonopoulos, K. A. Sound velocity and isentropic exponents for gases with different acentric factors by using the Redlich–Kwong–Soave equation of state. *Acta Mech.*, 1987, **66**, 177–189

7 Nelson, L. C. and Obert, E. F. Generalized *p-v-T* properties of gases. *Trans. ASME*, 1954, **76**, 1057

8 Cravalho, E. G. and Smith, J. L. *Engineering Thermodynamics*, Pitman Publishing Inc., London, 1981

9 Baehr, H. D. and Schwier, K. *Die Thermodynamischen Eigenschaften der Luft*, Springer-Verlag, Berlin, 1961

10 Soave, G. Equilibrium constants from a modified Redlich–Kwong equation of state. *Chem. Eng. Sci.*, 1972, **72**, 1197–1203

11 Soave, G. Rigorous and simplified procedures for determining the pure-component parameters in the Redlich–Kwong–Soave equation of state. *Chem. Eng. Sci.*, 1980, **35**, 1725–1729

12 Kouremenos, D. A. and Antonopoulos, K. A. Generalized and exact solutions for oblique shock waves of real gases with application to real air. *Int. J. Heat Fluid Flow*, 1989, **10**(4), 328–333